

Fourier's Law for a Granular Fluid

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(Dated: February 1, 2008)

Abstract

Newton's viscosity law for the momentum flux and Fourier's law for the heat flux define Navier-Stokes hydrodynamics for a simple, one component fluid. There is ample evidence that a hydrodynamic description applies as well to a mesoscopic granular fluid with the same form for Newton's viscosity law. However, theory predicts a qualitative difference for Fourier's law with an additional contribution from density gradients even at uniform temperature. The reasons for the absence of such terms for normal fluids are indicated, and a related microscopic explanation for their existence in granular fluids is presented.

PACS numbers:

I. INTRODUCTION

Granular fluids are of increasing interest to the Chemical Engineering, Physics, and Materials Sciences communities, for different and complementary reasons. On the practical side are industrial issues of agricultural, pharmaceutical, and chemical significance for packing and transport of grains. More recently, the planned expeditions to the moon and Mars require an understanding of the surface regolith, a new form of granular matter. In spite of the growing phenomenology for granular fluids, the fundamental descriptions for the simplest states remain subject to question. An important class of questions involve the form and conditions for a hydrodynamic description [1, 2, 3, 4, 5]. Here, one new feature of the granular Navier-Stokes hydrodynamic representation is addressed.

There are two peculiarities of the Navier-Stokes equations for granular fluids relative to those for normal fluids. One is a dissipative source of energy due to the inelasticity of collisions among the grains, easily understood as a consequence of the mesoscopic nature of the constituent particles. More puzzling is the modification of Fourier's law, with an additional contribution to the heat flux \mathbf{s} due to a density gradient

$$\mathbf{s} = -\lambda \nabla T - \mu \nabla n. \quad (1)$$

Here λ is the thermal conductivity and μ is the new transport coefficient characterizing heat flow in an isothermal, inhomogeneous fluid. Although physical interpretations of this latter effect have been given [6], it is instructive to associate it with the fundamental differences between normal and granular fluids. At the continuum level fluid symmetry alone leads to the general form (1) in both cases. Therefore, to better understand the presence of the density gradient contribution for granular fluids, it is useful to ask why it is absent for normal fluids.

This question is answered here at the level of the Boltzmann kinetic equation for a gas and using the more general statistical mechanics of linear response. In fact, a definitive conclusion requires information beyond continuum mechanics from the microscopic basis for hydrodynamics. The hydrodynamic fields are averages of the densities associated with the global invariants of mass, energy, and momentum. The hydrodynamic equations are then obtained from representations of the corresponding fluxes in the average microscopic conservation laws for these densities. There are two central observations here that are responsible for $\mu = 0$ in normal fluids. The first is that the mass flux is equal to the

momentum density. The second is that the reference state (equilibrium or local equilibrium) is entirely determined by the invariants. The first implies that there is no dissipation in the continuity equation for the mass density. The second imposes important constraints on the representation for transport coefficients in terms of correlation functions for the fluxes. The most compact version of these is Onsager's theorem [7], which is discussed next. Then more explicit realizations of these correlation functions are obtained from low density kinetic theory and formally exact statistical mechanics.

It should be emphasized that the existence of the coefficient μ is not in question, only its origin and interpretation. The coefficient has been determined for idealized isolated cooling granular gases [8], and measured in both simulations [9] and experiments [10] for shaken fluids in a gravitational field

It is a pleasure to dedicate this work to Keith Gubbins whose contributions to the kinetic theory and statistical mechanics of fluids have influenced me for more than thirty years. He has been an exceptional combination of mentor, colleague, role model, and friend to me and many others.

II. A CONSEQUENCE OF ONSAGER'S THEOREM

As noted above, the Navier-Stokes conditions of fluid symmetry and small spatial gradients do not constrain the "constitutive equation" for the heat flux beyond the form (1). However, in 1931 Onsager made a seminal observation relating the transport coefficients of a normal fluid to the underlying statistical mechanics of the fluid at equilibrium [7]. The Onsager regression hypothesis states that on long space and time scales the decay of spontaneous fluctuations in an equilibrium fluid is governed by the same laws as for nonequilibrium states displaced slightly from equilibrium. This is effectively what has become formalized as linear response theory. The symmetry property of these equilibrium fluctuations resulting from microscopic time reversal invariance of the dynamics leads to relations among the transport coefficients. These are the familiar Onsager relations for a fluid mixture. However, it is less well recognized that these same relations for a one component fluid imply that $\mu = 0$ in Eq. (1) [11, 12]

To show this explicitly, consider the exact macroscopic balance equations for the number

density $n(\mathbf{r}, t)$, energy density $e(\mathbf{r}, t)$, and the momentum density $\mathbf{g}(\mathbf{r}, t)$

$$\partial_t n(\mathbf{r}, t) + m^{-1} \nabla \cdot \mathbf{g}(\mathbf{r}, t) = 0, \quad (2)$$

$$\partial_t e(\mathbf{r}, t) + \nabla \cdot \mathbf{s}(\mathbf{r}, t) = w(\mathbf{r}, t), \quad (3)$$

$$\partial_t g_i(\mathbf{r}, t) + \partial_j t_{ij}(\mathbf{r}, t) = 0, \quad (4)$$

where $\mathbf{g}(\mathbf{r}, t)$, $\mathbf{s}(\mathbf{r}, t)$, and $t_{ij}(\mathbf{r}, t)$ are the associated mass, energy, and momentum fluxes and m is the mass. Here, it has already been assumed that the flux of mass is the same as the momentum density. In fact, this requires proof from the underlying microscopic balance equations. Also, $w(\mathbf{r}, t)$ is an energy source term that could be due to an external force doing work on the fluid, or the internal collisional energy loss of a granular fluid. The fluxes have a contribution due to convection and a purely dissipative contribution that occurs in the local rest frame at each point of the fluid. This separation is easily identified by a local Galilean transformation with the results [11]

$$\mathbf{g}(\mathbf{r}, t) \equiv mn(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t), \quad (5)$$

$$s_i(\mathbf{r}, t) = (e(\mathbf{r}, t) + p(\mathbf{r}, t)) u_i(\mathbf{r}, t) + t_{ij}^*(\mathbf{r}, t) u_j(\mathbf{r}, t) + s_i^*(\mathbf{r}, t) \quad (6)$$

$$t_{ij}(\mathbf{r}, t) = p(\mathbf{r}, t) \delta_{ij} + mn(\mathbf{r}, t) u_i(\mathbf{r}, t) u_j(\mathbf{r}, t) + t_{ij}^*(\mathbf{r}, t). \quad (7)$$

Equation (5) defines the local flow velocity $\mathbf{u}(\mathbf{r}, t)$, while (6) and (7) define the irreversible energy flux $s_i^*(\mathbf{r}, t)$ and momentum flux $t_{ij}^*(\mathbf{r}, t)$ in the rest frame. Finally, $p(\mathbf{r}, t)$ is the hydrostatic pressure which must be specified as a function of the local energy and density.

The set of exact equations (2)-(7) become a closed set of hydrodynamic equations once the "constitutive equations" for $s_i^*(\mathbf{r}, t)$ and $t_{ij}^*(\mathbf{r}, t)$ are given in terms of the number, energy, and momentum densities. In practice, it is useful to introduce a conjugate set of variables through the change of variables

$$\chi_\alpha(\mathbf{r}, t) \equiv -\frac{\partial s(\{y_\beta(\mathbf{r}, t)\})}{\partial y_\alpha(\mathbf{r}, t)}, \quad y_\alpha(\mathbf{r}, t) \Leftrightarrow (n(\mathbf{r}, t), e(\mathbf{r}, t), \mathbf{g}(\mathbf{r}, t)). \quad (8)$$

Here $s(\{y_\beta(\mathbf{r}, t)\})$ is the equilibrium entropy density for the fluid as a function of the number, energy, and momentum densities at each point of the fluid. For example, it follows that $\chi_2 = -1/T$ which defines the temperature as a function of the density and energy. Note that although this entropy function is defined for an equilibrium fluid, it is used here simply

as the mathematical generator of a change of variables. As such, it applies even to states far from equilibrium and also to the granular fluid.

For states near uniform equilibrium, the leading contributions to $\mathbf{s}^*(\mathbf{r}, t)$ and $t_{ij}^*(\mathbf{r}, t)$ are linear in gradients of the $\{y_\alpha\}$, or equivalently, gradients of $\{\chi_\alpha\}$. The resulting equations (2)-(4) are the Navier-Stokes order hydrodynamics

$$\partial_t y_\alpha(\mathbf{r}, t) + \nabla \cdot \mathbf{j}_\alpha(\mathbf{r}, t) = w_\alpha, \quad (9)$$

where the fluxes are given by

$$\mathbf{j}_\alpha(\mathbf{r}, t) = \mathbf{j}_\alpha^{(0)}(\mathbf{r}, t) + \mathbf{j}_\alpha^*(\mathbf{r}, t), \quad \mathbf{j}_\alpha(\mathbf{r}, t) \Leftrightarrow (\mathbf{g}(\mathbf{r}, t), \mathbf{s}(\mathbf{r}, t), t_{ij}(\mathbf{r}, t)). \quad (10)$$

Here, $\mathbf{j}_\alpha^{(0)}$ are the Euler order contributions identified from (6) and (7), and \mathbf{j}_α^* are the irreversible contributions. At Navier-Stokes order the latter are given by the linear constitutive equations [11, 12]

$$\mathbf{j}_\alpha^*(\mathbf{r}, t) = - \sum_\beta L_{\alpha\beta} (\{y_\beta(\mathbf{r}, t)\}) : \frac{\partial \chi_\beta(\mathbf{r}, t)}{\partial \mathbf{r}}. \quad (11)$$

The energy flux in this approximation is the generalized Fourier's law (1) and the momentum flux is the usual Newton's viscosity law.

Up to this point only the macroscopic conservation laws, gradient expansion near the reference homogeneous state, and fluid symmetry have been used. For a normal fluid Onsager's regression hypothesis allows identification of the transport coefficients $L_{\alpha\beta}$ in the form

$$L_{\alpha\beta} = \int_0^\infty dt C_{\alpha\beta}(t), \quad (12)$$

where $C_{\alpha\beta}(t)$ is an equilibrium time correlation function for two microscopic fluxes corresponding to the macroscopic \mathbf{j}_α . These fluxes have a definite parity under the transformation $t \rightarrow -t$, and reversal of all particle velocities $\{\mathbf{v}_i\} \rightarrow \{-\mathbf{v}_i\}$. Since both the dynamics and the equilibrium ensemble are invariant under this transformation it follows that

$$C_{\alpha\beta}(t) = \tau_\alpha \tau_\beta C_{\beta\alpha}(t), \quad (13)$$

where $\tau_\alpha = \pm 1$, depending on the parity of the associated flux. Therefore $L_{\alpha\beta}$ has the symmetry

$$L_{\alpha\beta} = \tau_\alpha \tau_\beta L_{\beta\alpha}. \quad (14)$$

This is Onsager's theorem.

The coefficient μ in (1) is identified as

$$\mu = L_{21} \frac{\partial \chi_1}{\partial n} |_T. \quad (15)$$

An important observation now is that the macroscopic conservation law for the number density, the continuity equation, has no dissipative contribution, $L_{1\beta} = 0$, since the mass flux is equal to the momentum density. Consequently, Onsager's theorem gives

$$L_{\beta 1} = L_{1\beta} = 0, \quad (16)$$

and Fourier's law becomes

$$\mathbf{s}^*(\mathbf{r}, t) = -\lambda (\{y_\beta(\mathbf{r}, t)\}) \nabla T(\mathbf{r}, t), \quad \lambda = L_{22} T^{-2}. \quad (17)$$

(There is no coupling to the velocity gradients, $L_{\alpha 3} = 0$, from fluid symmetry). Thus, $\mu = 0$ for a normal fluid as a consequence of the fact that the number flux in the continuity equation has no dissipative contribution, and the symmetry of the correlation functions resulting from the fact that the reference state is a function of the invariants. These properties are demonstrated more explicitly in the further microscopic elaboration below.

The above analysis holds as well for granular fluids, except for Onsager's theorem. The condition $L_{1\beta} = 0$ still applies but the key symmetry, (13), rests on both the equilibrium Gibbs reference state and the invariance of the dynamics for the system. Neither the homogeneous reference state nor the dynamics of a granular fluid have these symmetries, so neither the usual form of Onsager's theorem nor the conclusion that $\mu = 0$ can be extended to granular fluids. This failure of Onsager's theorem occurs as well for normal fluids in nonequilibrium stationary states. However, it is possible that some other symmetry could apply to enforce the usual form of Fourier's law for granular fluids. Thus, it is important to look in more detail at the mesoscopic (kinetic theory) and microscopic (statistical mechanics) representations of this constitutive relation. This is the objective of the next two sections.

III. KINETIC THEORY

The simplest fluid is a low density gas for which the appropriate kinetic theory is given by the Boltzmann equation. The usual derivations of this equation for a normal gas can

be extended to the granular gas as well, to account for binary collisions that are inelastic [14, 15, 16]. It provides an important testing ground for the derivation of hydrodynamics and for exploration of conceptual issues as well. The application to a normal gas is considered first, and then its extension to a granular gas.

A. Normal gas

The Boltzmann equation for the one particle reduced distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is

$$(\partial_t + \mathbf{v} \cdot \nabla) f = \mathcal{C}[f, f]. \quad (18)$$

where $\mathcal{C}[f, f]$ describes uncorrelated elastic binary collisions [11]. The notation denotes that $\mathcal{C}[f, f]$ is a bilinear functional of f . Its detailed form will not be required here, only the fact that particle number, energy, and momentum are conserved as represented by the properties

$$\int d\mathbf{v} a_\alpha \mathcal{C}[f, f] = 0, \quad a_\alpha(\mathbf{v}) \leftrightarrow \left(1, \frac{1}{2}mv^2, m\mathbf{v}\right). \quad (19)$$

The set of functions $\{a_\alpha\}$ are known as the summational invariants. Their averages are also the hydrodynamic fields of the last section

$$y_\alpha(\mathbf{r}, t) = \int d\mathbf{v} a_\alpha(\mathbf{v}) f(\mathbf{r}, \mathbf{v}, t). \quad (20)$$

The macroscopic balance equations (2)-(4), or in the equivalent compact form (9), follow from this definition by differentiation with respect to time and application of the Boltzmann equation. In this way the fluxes are identified as

$$\mathbf{j}_\alpha(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{b}_\alpha(\mathbf{v}) f(\mathbf{r}, \mathbf{v}, t), \quad \mathbf{b}_\alpha(\mathbf{v}) = \mathbf{v} a_\alpha(\mathbf{v}). \quad (21)$$

Hydrodynamic equations result from these exact consequences of Boltzmann's equation when the solution f approaches a "normal" form on some length and time scale, expected to be long compared to the mean free space and time [5]. A normal distribution is one for which all space and time dependence occurs through the hydrodynamic fields

$$f(\mathbf{r}, \mathbf{v}, t) \rightarrow f(\mathbf{v} | \{y_\alpha\}). \quad (22)$$

The notation $f(\mathbf{v} | \{y_\alpha\})$ indicates a *functional* of the fields $y_\alpha(\mathbf{r}, t)$ throughout the system (equivalently, and for practical purposes, it is a *function* of the fields and all their derivatives

at the point of interest). An example of a normal distribution is the local equilibrium distribution

$$f_\ell(V) \equiv n \left(\frac{m}{2\pi T} \right)^{3/2} \exp(-mV^2/2T). \quad (23)$$

where $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$ and in all of the following, units are used such that Boltzmann's constant $k_B = 1$. This distribution is parameterized by five fields n , T , and \mathbf{u} which, as the notation suggests, are chosen to be the *same* as the hydrodynamic fields defined in (20) and (8). This is expressed by the condition

$$\int d\mathbf{v} a_\alpha (f - f_\ell) = 0. \quad (24)$$

The local equilibrium distribution function depends on the fields, but not their gradients. In fact, it is a solution to the Boltzmann equation to zeroth order in the gradients, as follows from the second important property of the collision operator

$$\mathcal{C}[f_\ell, f_\ell] = 0. \quad (25)$$

More generally, the normal solution to the Boltzmann equation can be constructed as an expansion in the gradients with f_ℓ as the leading order contribution. This is done by the familiar Chapman-Enskog procedure [11] and is carried out in the Appendix.

It is useful to provide a geometrical representation of the normal solution. First, define a set of functions "conjugate" to the $\{a_\alpha\}$ by

$$\psi_\nu = \frac{\partial f_\ell}{\partial y_\nu}, \quad \int d\mathbf{v} a_\alpha \psi_\nu = \delta_{\alpha\nu}. \quad (26)$$

The second equality follows from (20) and (24) and shows the sense in which $\{a_\alpha\}$ and $\{\psi_\alpha\}$ form a biorthogonal set. Next, define the projection operator whose action on an arbitrary distribution function h is

$$\mathcal{P}h = \psi_\nu \int d\mathbf{v} a_\nu h. \quad (27)$$

It is easily verified that \mathcal{P} has the property of a projection operator $\mathcal{P}^2 = \mathcal{P}$. The utility of these definitions is the decomposition of f into its local equilibrium distribution plus a remainder that is in the orthogonal subspace defined by $\mathcal{Q} = 1 - \mathcal{P}$

$$f = f_\ell + \delta f = f_\ell + \mathcal{P}\delta f + \mathcal{Q}\delta f = f_\ell + \mathcal{Q}\delta f. \quad (28)$$

The last equality follows from (24), $\mathcal{P}\delta f = 0$. Since f_ℓ is a function of the fields, and not their gradients, all contributions to a normal solution due to gradients must come from

$\delta f = \mathcal{Q}\delta f$. Furthermore, since f_ℓ is a solution to the Boltzmann equation at zeroth order in the gradients (see (25)), $\delta f = \mathcal{Q}\delta f$ is at least of first order. This allows a decomposition of the fluxes (21) into Euler and irreversible contributions as in the previous section with the identifications

$$\mathbf{j}_\alpha^{(0)}(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{b}_\alpha f_\ell, \quad \mathbf{j}_\alpha^*(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{b}_\alpha \mathcal{Q}\delta f. \quad (29)$$

An immediate consequence of the property $\mathbf{b}_1(\mathbf{V}) = \mathbf{a}_3(\mathbf{V})$ is $\mathbf{j}_1^* = 0$, resulting in the continuity equation for the number density.

The Boltzmann equation determines the detailed form for δf . Assuming a normal form, the Boltzmann equation to first order in the gradients becomes (see Appendix)

$$L\delta f = -(\mathcal{Q}\gamma_\alpha) \cdot \nabla_{\mathbf{r}} \chi_\alpha, \quad (30)$$

where L is the linear Boltzmann collision operator

$$Lh \equiv -\mathcal{C}[f_\ell, h] - \mathcal{C}[h, f_\ell]. \quad (31)$$

A transformation to the conjugate variables has been made using

$$\nabla_{\mathbf{r}} y_\alpha = g_{\alpha\nu}^{-1} \nabla_{\mathbf{r}} \chi_\nu, \quad g_{\nu\alpha} = g_{\alpha\nu} = \frac{\partial \chi_\alpha}{\partial y_\nu}, \quad (32)$$

and $\gamma_\alpha(\mathbf{v})$ are the conjugate fluxes are defined by

$$\gamma_\alpha(\mathbf{v}) = \mathbf{v} \psi_\nu(\mathbf{v}) g_{\nu\alpha}^{-1} = \mathbf{v} \frac{\partial f_\ell(\mathbf{v})}{\partial \chi_\alpha} \quad (33)$$

The appearance of the orthogonal projection \mathcal{Q} on the right side of (30) assures that solutions to this equation exist. This is given by the Fredholm alternative for such linear inhomogeneous equations which states that the right side must be orthogonal to the null space for the adjoint of \mathcal{L} , in this case given by $\mathcal{P}\mathcal{L} = 0$. The formal solution can be written

$$\delta f = -\mathcal{Q} \int_0^\infty dt e^{-Lt} \mathcal{Q}\gamma_\alpha(\mathbf{V}) \cdot \nabla_{\mathbf{r}} \chi_\alpha, \quad (34)$$

In general, an arbitrary solution to the homogeneous equation $L\delta f = 0$ could be added to this, but the condition (24), $\mathcal{P}\delta f = 0$ excludes such terms.

Use of this formal solution in (29) gives the linear constitutive equations (11), and the transport matrix $L_{\alpha\beta}$ is identified as

$$L_{\alpha\beta} = \int_0^\infty d\tau C_{\alpha\beta}(\tau), \quad (35)$$

$$C_{\alpha\beta}(\tau) = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_\alpha) e^{-L\tau} \mathcal{Q} \gamma_\beta. \quad (36)$$

where \mathcal{Q}^\dagger is the adjoint of \mathcal{Q} . This makes explicit the low density form for the general phenomenological postulate. It is given in the form of a Green-Kubo expression, where the "time correlation function" $C_{\alpha\beta}(\tau)$ is a flux - conjugate flux correlation function. Further simplifications are possible by noting that both the conjugate densities ψ_ν and conjugate fluxes γ_ν can be written as linear combinations of the densities a_ν and \mathbf{b}_ν , respectively

$$\psi_\nu = f_\ell \sum_\sigma c_{\nu\sigma} a_\sigma, \quad \gamma_\nu = f_\ell \sum_\sigma c_{\nu\sigma} \mathbf{b}_\sigma. \quad (37)$$

Note that the (adjoint) projection operator \mathcal{Q}^\dagger implies that $C_{1\beta}(\tau) = 0$ since $\mathbf{b}_1 \propto \mathbf{a}_3$ (the mass flux is the momentum density) and \mathcal{Q}^\dagger projects orthogonal to the set of densities $\{a_\alpha\}$.

The μ coefficient vanishes for similar reasons. It is given by

$$\mu = \int_0^\infty d\tau C_{21}(\tau) \frac{\partial \chi_1}{\partial n} \Big|_T. \quad (38)$$

The conjugate flux γ_1 is proportional to the conjugate density ψ_3

$$\gamma_1 = \mathbf{v} \frac{\partial f_\ell}{\partial \chi_1} \Big|_T = -\frac{T}{mn} \frac{\partial \chi_1}{\partial n} \Big|_T \psi_3. \quad (39)$$

Since \mathcal{Q} projects orthogonal to the conjugate densities $\{\psi_\alpha\}$, i.e. $\mathcal{Q}\psi_3 = 0$, the projected flux $\mathcal{Q}\gamma_\beta$, the correlation function $C_{21}(t)$, and consequently μ all vanish

$$\mu = -\frac{T}{mn} \frac{\partial \chi_1}{\partial n} \Big|_T \int_0^\infty d\tau \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_\alpha)_2 e^{-L\tau} \mathcal{Q}\psi_3 = 0, \quad (40)$$

In summary, the correlation functions determining the transport coefficients involve fluxes projected orthogonal to associated densities. The fluxes associated with the density are themselves densities, and hence such correlation functions vanish. In particular those associated with dissipation in the continuity equation and the contribution to the heat flux from density gradients are zero.

B. Granular gas

The analysis for the granular Boltzmann equation proceeds in a similar way [17], and the details are given in Appendix A. The correlation functions for the transport matrix $L_{\alpha\beta}$ in

this more general case are

$$C_{\alpha\beta}(\tau) = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_\alpha) e^{-\mathcal{L}\tau} (e^{\Lambda\tau} \mathcal{Q}\boldsymbol{\gamma})_\beta. \quad (41)$$

$$\mathcal{L} \equiv \frac{1}{2}\xi \nabla_{\mathbf{v}} \cdot \mathbf{V} + L, \quad \xi \equiv -\frac{2}{dnT} w[f^{(0)}, f^{(0)}]. \quad (42)$$

The parameter ξ is called the cooling rate since it determines the relative rate of change of the temperature in the HCS due to inelastic collisions. The densities $\{a_\alpha\}$ and fluxes $\{\mathbf{b}_\alpha\}$ are the same as those in (19) and (21). The projection operator is given by (21) but with new conjugate densities $\{\psi_\alpha\}$ and fluxes $\{\boldsymbol{\gamma}_\alpha\}$ defined in terms of the reference local homogeneous cooling state (HCS) $f^{(0)}$ instead of the local equilibrium state f_ℓ

$$\psi_\nu = \frac{\partial f^{(0)}}{\partial y_\nu}, \quad \gamma_\alpha = \mathbf{v} \frac{\partial f^{(0)}(\mathbf{v})}{\partial \chi_\alpha}. \quad (43)$$

The reference HCS is the normal solution to the Boltzmann equation in the absence of gradients

$$\frac{1}{2}\xi \nabla_{\mathbf{v}} \cdot (\mathbf{V} f^{(0)}) = \mathcal{C}[f^{(0)}, f^{(0)}]. \quad (44)$$

The cooling rate ξ vanishes in the elastic limit and $f^{(0)} \rightarrow f_\ell$ in this limit.

Aside from these differences in the reference state for granular fluids, the generator for dynamics in (41) shows significant differences. The linearized operator L is now that associated with the Boltzmann collision operator for inelastic collisions. In addition, there is a velocity scaling operator $\frac{1}{2}\xi \nabla_{\mathbf{v}} \cdot \mathbf{V}$ (the derivative operates on everything to its right) which compensates for the cooling generated by the collisions, as illustrated in the exact balance of these effects in (44). Finally, there is the matrix Λ whose explicit form is given in (A22) and (A24) of the Appendix. The relevant point here is that its eigenvalues $(0, \frac{1}{2}\xi, -\frac{1}{2}\xi)$ are the same as the smallest eigenvalues of \mathcal{L} . Thus, the entire generator for the dynamics has a null space, just as for normal fluids, and the projection \mathcal{Q} assures that the fluxes are orthogonal to the invariants defining this null space. This is a necessary condition for the existence of the integral defining the transport matrix $L_{\alpha\beta}$.

It is seen that the condition for the continuity equation still holds $C_{1\beta}(\tau) = 0$, for the same reason as in normal fluids, $\mathcal{Q}^\dagger \mathbf{b}_1 \propto \mathcal{Q}^\dagger \mathbf{a}_3 = 0$. However, the related conditions for the transport coefficient μ to vanish are no longer satisfied in general. The relevant correlation function $C_{21}(\tau)$ is no longer determined by the single flux $\boldsymbol{\gamma}_1$ but is coupled to $\boldsymbol{\gamma}_2$ as well. Neither of these fluxes is simply proportional to the conjugate densities $\{\psi_\alpha\}$ and hence the

action of \mathcal{Q} on each is non zero. This difference occurs for granular fluids because $f^{(0)}$ is no longer determined entirely in terms of the $\{a_\alpha\}$ and hence there is no simple relationship between the sets $\{a_\alpha\}, \{\mathbf{b}_\alpha\}$ and $\{\psi_\alpha\}, \{\gamma_\alpha\}$ as expressed in (37). The explicit form for μ is obtained in Appendix A

$$\begin{aligned} \mu = & - \int_0^\infty d\tau \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_2) \left\{ e^{-(\mathcal{L}-\lambda_1)\tau} \mathcal{Q} \mathbf{v} \frac{\partial f^{(0)}}{\partial n} \Big|_T \right. \\ & \left. - 2 \frac{\partial e}{\partial n} \Big|_T (e^{-(\mathcal{L}-\lambda_2)\tau} - e^{-(\mathcal{L}-\lambda_1)\tau}) \mathcal{Q} \left(\mathbf{v} \frac{\partial f^{(0)}}{\partial e} \Big|_n \right) \right\}. \end{aligned} \quad (45)$$

The new conjugate densities $\{\psi_\alpha\}$ are invariants of the new dynamics generated by $(\mathcal{L} - \lambda_\alpha)$. The projection operators again assure that there is no contribution from the invariants which is a necessary condition for convergence of the τ integral. In general, however, neither $\mathbf{v} \partial f^{(0)} / \partial n \Big|_T$ nor $\mathbf{v} \partial f^{(0)} / \partial e \Big|_T$ is a linear combination of these invariants and the action of \mathcal{Q} on them does not vanish. Only in the elastic limit does $\mathcal{Q} \mathbf{v} \partial f^{(0)} / \partial n \Big|_T$ become proportional to $\mathcal{Q} \psi_3 = 0$. In this limit $\lambda_2 = \lambda_1 = 0$ and the coefficient of $\mathbf{v} \partial f^{(0)} / \partial e \Big|_T$ also vanishes, confirming $\mu = 0$ for a normal fluid.

IV. FORMAL LINEAR RESPONSE

The analysis of hydrodynamics from kinetic theory can be generalized by the formal application of nonequilibrium statistical mechanics to granular fluids. The details are described in references [18, 19]. The starting point is the Liouville equation for the N particle phase space density $\rho(\Gamma, t)$, where $\Gamma \equiv \{\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{v}_1, \dots, \mathbf{v}_N\}$ denotes a point in the $6N$ dimensional phase space [20]. First, a homogeneous normal solution to the Liouville equation is $\rho_h(\Gamma)$ identified, representing the homogeneous cooling state (HCS) for an isolated system

$$\overline{\mathcal{L}} \rho_h = 0, \quad \overline{\mathcal{L}} X \equiv \frac{1}{2} \zeta_h \sum_{i=1}^N \nabla_{\mathbf{v}_i} \cdot (\mathbf{V}_i X) + \overline{L} X. \quad (46)$$

Here \overline{L} is the Liouville operator for N hard inelastic spheres and ζ_h is the associated cooling rate in the HCS. In addition, there is a scaling operator $\frac{1}{2} \zeta_h \nabla_{\mathbf{v}_i} \cdot \mathbf{V}_i$ for each particle. Clearly, $\overline{\mathcal{L}}$ is the N particle generalization of the kinetic theory generator \mathcal{L} of (42). Next, small spatial perturbations of this state are induced through an associated local HCS $\rho_{\ell h}$. The response of the hydrodynamic fields at a later time due to these initial perturbations is characterized by response functions, represented as time correlation functions composed of

the N particle phase functions for the hydrodynamic fields (corresponding to the a_α of the last section) and the functional derivatives of ρ_{lh} with respect to the conjugate fields $\{\chi_\alpha\}$ (corresponding to the ψ_α of the last section). Since these response functions must exhibit hydrodynamic excitations at long times and long wavelengths, the transport matrix $L_{\alpha\beta}$ can be identified. It has the representation in terms of time correlation functions again, as in (35) but now the results are formally exact without the restrictions of the kinetic theory.

For the purposes here it is sufficient to display only the final result for the coefficient μ . It has a form similar to that of (38)

$$\mu = \mu_0 + \int_0^\infty d\tau C_{21}(\tau) \frac{\partial \chi_1}{\partial n} \Big|_T. \quad (47)$$

There is an additional term μ_0 that does not have the form of a time integral of a correlation function, and is due to both the singular hard sphere dynamics and the dissipation. It vanishes in the low density limit, and so does not appear at the level of the Boltzmann equation. It is given by

$$\mu_0 = \frac{e}{dT} V^{-1} \int d\Gamma \mathbf{S}(\Gamma) \cdot \mathbf{M}(\Gamma). \quad (48)$$

where $\mathbf{S}(\Gamma)$ is the volume integrated phase function for the energy flux (corresponding to \mathbf{b}_2 in the kinetic theory analysis), and $\mathbf{M}(\Gamma)$ is the space moment for the functional derivative of ρ_{lh} with respect to density

$$\mathbf{M} = \int d\mathbf{r} \mathbf{r} \left(\frac{\delta \rho_{lh}}{\delta n(\mathbf{r})} \Big|_T \right)_{\delta y=0}. \quad (49)$$

It is easily seen that $\mu_0 \rightarrow 0$ in the elastic limit, for which ρ_{lh} becomes the corresponding equilibrium ensemble. Otherwise it is non-zero. The correlation function in (47) has a form similar to that of (41)

$$C_{21}(\tau) \frac{\partial \chi_1}{\partial n} \Big|_T = V^{-1} \int d\Gamma (\mathcal{Q}^\dagger \mathbf{S}) e^{-\bar{\mathcal{L}}\tau} (e^{\Lambda\tau} \mathcal{Q} \Upsilon)_1 \quad (50)$$

where the conjugate flux is

$$\Upsilon_\alpha \equiv -(\bar{\mathcal{L}} - \Lambda) \mathbf{M}_\alpha \quad (51)$$

Here \mathbf{M}_α is the space moment of the functional derivative of ρ_{lh} with respect to y_α holding the other $\{y_\beta\}$ constant, and evaluated at the HCS.

The projection operator \mathcal{Q} projects orthogonal to the invariants of the dynamics generated by $(\bar{\mathcal{L}} - \Lambda)$. Consider now the elastic limit for which ρ_{lh} becomes a local equilibrium

ensemble ρ_{le} . To be specific, let that be the local grand canonical ensemble. The correlation function simplifies to

$$C_{21}(\tau) \frac{\partial \chi_1}{\partial n} |_T \rightarrow -V^{-1} \int d\Gamma (\mathcal{Q}^\dagger \mathbf{S}) e^{-\bar{L}\tau} \mathcal{Q} \bar{L} \mathbf{M}, \quad (52)$$

$$\mathbf{M} \rightarrow -\rho_e m \int d\mathbf{r} \mathbf{r} \int d\mathbf{r}' \left(\frac{\delta \chi_1(\mathbf{r}')}{\delta n(\mathbf{r})} |_T \right)_{\delta y=0} \hat{n}(\mathbf{r}'), \quad (53)$$

where ν is the activity, ρ_e is the strict equilibrium ensemble, and $\hat{n}(\mathbf{r})$ is the phase function corresponding to the number density. Then using $\bar{L}\hat{n}(\mathbf{r}) = -m^{-1} \nabla \cdot \hat{\mathbf{g}}(\mathbf{r})$, where $\hat{\mathbf{g}}(\mathbf{r})$ is the phase function representing the momentum density, the flux $\bar{L}\mathbf{M}$ is found to be proportional to the total momentum \mathbf{P}

$$\begin{aligned} \bar{L}\mathbf{M} &= \rho_e m \int d\mathbf{r} \mathbf{r} \int d\mathbf{r}' \left(\frac{\delta \chi_1(\mathbf{r}')}{\delta n(\mathbf{r})} |_T \right)_{\delta y=0} \bar{L}n(\mathbf{r}') \\ &= -\rho_e \int d\mathbf{r} \mathbf{r} \int d\mathbf{r}' \left(\frac{\delta \chi_1(\mathbf{r}')}{\delta n(\mathbf{r})} |_T \right)_{\delta y=0} \nabla' \cdot \mathbf{g}(\mathbf{r}') \\ &= \rho_e \frac{\partial \chi_1}{\partial n} |_T \mathbf{P}. \end{aligned} \quad (54)$$

Consequently, $\mathcal{Q}\bar{L}\mathbf{M} = \mathbf{0}$ and $C_{21}(\tau) = 0$. It was already noted that μ_0 vanishes in the elastic limit, so $\mu = 0$ as well and the usual form of Fourier's law is recovered.

More generally, for inelastic collisions $\mu_0 \neq 0$ (except at low density) and none of the fluxes $\mathbf{\Upsilon}$ are simply proportional to the invariants. Consequently, $\mathcal{Q}\mathbf{\Upsilon} \neq 0$ and $\mu \neq 0$.

V. DISCUSSION

The theoretical "discovery" that Fourier's law for a granular fluid has an additional term proportional to the density gradient provides an interesting qualitative difference from normal fluids. There has been much discussion about this additional term and attempts to detect it in simulations or experiments. The theoretical analysis here provides a different perspective, in hindsight, that the surprising nature of this difference between Navier-Stokes hydrodynamics for normal and granular fluids is the *absence* of this term in the former case rather than its presence for the latter case. This circumstance is similar to the discovery of generic long range correlations in nonequilibrium states, absent at equilibrium [21]. Both provide examples of the very special balance of competing effects for the equilibrium state, in contrast to the qualitatively different behavior for generic nonequilibrium states.

The vanishing of the transport coefficient for the contribution to the heat flux from a density gradient follows from the simple structure of the local conservation law for the number density, and the characterization of the equilibrium state in terms of the dynamical invariants. The number conservation law relates the density to its flux. However, this flux is itself proportional to a density for one of the invariants - the momentum. The transport coefficients are time integrals of correlation functions composed of these fluxes and corresponding "conjugate" fluxes generated from the equilibrium ensemble. Convergence of the time integrals requires that there be no time independent parts to these correlation functions. Thus, the correlation functions are constructed from those parts of the fluxes that have their invariant parts subtracted (or projected) out. As the number flux is itself an invariant all transport processes coupling to the number flux will therefore have zero transport coefficients. This leads directly to the vanishing of all dissipative contributions to the continuity equation. In addition, since the equilibrium ensemble is a function of the invariants it results that the conjugate fluxes are linear combinations of the fluxes in the conservation laws. Thus any contributions from the number flux, itself a density, has no remainder once its invariant parts are subtracted out. This leads to the vanishing of the coefficient μ in Fourier's law.

The simple structure of the conservation law for number density does not depend on the state of the system considered. Thus, there are no dissipative contributions to the number flux for any state, equilibrium or nonequilibrium. In contrast, the conjugate fluxes depend in detail on the reference state about which hydrodynamic excitations are being considered. The simple relationship of the conjugate fluxes to the fluxes in the conservation laws for equilibrium states cannot be expected more generally for any nonequilibrium reference state. This is the case for both normal and granular fluids. An example of the former is the hydrodynamic excitations about uniform shear flow [13] where Fourier's law in the form of Eq. (1) applies with $\mu \neq 0$.

In closing, it may be useful to display concrete expressions for λ and μ in a granular gas obtained from an approximate evaluation of the expression given above from the Boltzmann equation [22] for $d = 3$

$$\lambda = \frac{nT}{m} \frac{5}{2(\nu - 2\xi)}, \quad \mu = \frac{T}{n} \lambda \frac{2\xi}{2\nu - 3\xi}$$

where $\xi = 5\nu_0(1 - \alpha^2)/12$ is the cooling rate of (44) calculated in this same approximation,

$\nu = \nu_0 (1 + \alpha) (49 - 33\alpha)/48$, and $\nu_0 = 16n\sigma^2 (\pi T/m)^{1/2}/5$ is an average collision rate. In the nearly elastic limit $\lambda \rightarrow 15nT/4m\nu_0$ and $\mu \rightarrow (1 - \alpha) 75T^2/16m\nu_0$. The simulations of references [9] and [8] confirm the more general theoretical prediction over a wide range of values for the restitution coefficient α .

VI. ACKNOWLEDGEMENTS

Comments from A. Baskaran, Syracuse University, J. J. Brey, Universidad de Sevilla, and A. Santos, Universidad de Extremadura, are gratefully acknowledged.

APPENDIX A: NORMAL SOLUTION TO BOLTZMANN EQUATION

In this Appendix the normal solution to the Boltzmann equation for a granular gas is obtained up through first order in the gradients [17],

$$f = f^{(0)} + \mathbf{F}_\alpha \cdot \nabla_{\mathbf{r}} y_\alpha + \dots \quad (\text{A1})$$

The leading order term $f^{(0)}$ and coefficient \mathbf{F}_α are functions of the actual hydrodynamic fields $\{y_a\}$. As a normal solution obeys a condition analogous to (24)

$$\int d\mathbf{v} a_\alpha (f - f^{(0)}) = 0. \quad (\text{A2})$$

Only the case of hard sphere interactions is considered, so there is no internal energy scale. Then, from dimensional analysis, they have the forms

$$f^{(0)} \equiv n \left(\frac{m}{2\pi T} \right)^{3/2} f^{(0)*}(V^*), \quad , \quad \mathbf{V}^* = \sqrt{\frac{m}{2T}} (\mathbf{v} - \mathbf{U}), \quad (\text{A3})$$

$$\mathbf{F}_1 = \left(\frac{m}{2T} \right)^{3/2} \mathbf{F}_1^*(\mathbf{V}^*), \quad \mathbf{F}_2 = \frac{2}{dT} \left(\frac{m}{2T} \right)^{3/2} \mathbf{F}_\alpha^*(\mathbf{V}^*), \quad \mathbf{F}_3 = \frac{1}{m} \left(\frac{m}{2T} \right)^2 \mathbf{F}_\alpha^*(\mathbf{V}^*). \quad (\text{A4})$$

The temperature T for hard spheres is related to the energy by $e = \frac{3}{2}nT + \frac{1}{2}mnU^2$, and the asterisk denotes a dimensionless quantity. Substitution of (A1) into the Boltzmann equation gives

$$\left(\frac{\partial f^{(0)}}{\partial y_\sigma} + \frac{\partial \mathbf{F}_\alpha}{\partial y_\sigma} \cdot \nabla_{\mathbf{r}} y_\alpha + \mathbf{F}_\sigma \cdot \nabla + \dots \right) (\partial_t y_\sigma + \mathbf{v} \cdot \nabla_{\mathbf{r}} y_\sigma) = \mathcal{C}[f, f], \quad (\text{A5})$$

and using the macroscopic balance equations (9) the time derivative can be expressed in terms of the gradients

$$\left(\frac{\partial f^{(0)}}{\partial y_\sigma} + \frac{\partial \mathbf{F}_\alpha}{\partial y_\sigma} \cdot \nabla_{\mathbf{r}} y_\alpha + \mathbf{F}_\sigma \cdot \nabla + \dots \right) (w_\sigma - \nabla \cdot \mathbf{j}_\sigma + \mathbf{v} \cdot \nabla_{\mathbf{r}} y_\sigma) = \mathcal{C}[f, f]. \quad (\text{A6})$$

To zeroth order in the gradients this equation determines $f^{(0)}$

$$\frac{\partial f^{(0)}}{\partial y_\alpha} w_\alpha [f^{(0)}, f^{(0)}] = \mathcal{C}[f^{(0)}, f^{(0)}], \quad (\text{A7})$$

where the bilinear functional dependence of the energy loss w_σ has been made explicit to show that here it is evaluated to lowest order. Since the source w_α occurs only in the energy equation the left side of this equation can be made more explicit

$$\frac{\partial f^{(0)}}{\partial y_\sigma} w_\sigma [f^{(0)}, f^{(0)}] = \frac{\partial f^{(0)}}{\partial e} w [f^{(0)}] = \frac{\partial f^{(0)}}{\partial T} \frac{2}{3n} w [f^{(0)}, f^{(0)}] = \frac{1}{2} \xi (d + \mathbf{V} \cdot \nabla) f^{(0)}, \quad (\text{A8})$$

and the cooling rate $\xi \equiv -2w [f^{(0)}, f^{(0)}] / 3nT$ has been introduced. Equation (A7) for $f^{(0)}$ becomes

$$\frac{1}{2} \xi \nabla_{\mathbf{V}} \cdot (\mathbf{V} f^{(0)}) = \mathcal{C}[f^{(0)}, f^{(0)}]. \quad (\text{A9})$$

Although simple analytic forms for the solution to this equation have not yet been found, its behavior for small and large velocities is known, and good approximations exist more generally. Furthermore, it has been studied numerically using Direct Simulation Monte Carlo. For our purposes, therefore, it can be considered as known. In the elastic limit the solution to $\mathcal{C}[f^{(0)}, f^{(0)}] = 0$ is the Maxwellian (23).

To first order in the gradients Eq. (A6) determines \mathbf{F}_α as the solution to

$$\left[L\mathbf{F}_\alpha + \left(w \frac{\partial \mathbf{F}_\alpha}{\partial e} + \frac{\partial w}{\partial y_\alpha} \mathbf{F}_2 \right) \right] \cdot \nabla_{\mathbf{r}} y_\alpha = - \frac{\partial f^{(0)}}{\partial y_\alpha} (\mathbf{v} \cdot \nabla_{\mathbf{r}} y_\alpha - \nabla_{\mathbf{r}} \cdot \mathbf{j}_\alpha^{(0)}) \quad (\text{A10})$$

$$Lh \equiv -\mathcal{C}[f_\ell, h] - \mathcal{C}[h, f_\ell]. \quad (\text{A11})$$

It is understood here that $w = w_\alpha [f^{(0)}, f^{(0)}]$. The Euler order flux $\mathbf{j}_\alpha^{(0)}$ is defined in terms of the lowest order distribution as in (29)

$$\mathbf{j}_\alpha^{(0)}(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{b}_\alpha(\mathbf{v}) f^{(0)}(\mathbf{r}, \mathbf{v}, t). \quad (\text{A12})$$

It follows directly that the second term on the right side of (A10) can be written

$$\frac{\partial f^{(0)}}{\partial y_\alpha} \nabla_{\mathbf{r}} \cdot \mathbf{j}_\alpha^{(0)}(\mathbf{r}, t) = \psi_\alpha \int d\mathbf{v} \mathbf{a}_\alpha(\mathbf{v}) \mathbf{v} \psi_\alpha(\mathbf{v}) \cdot \nabla_{\mathbf{r}} y_\alpha = \mathcal{P}(\mathbf{v} \psi_\alpha(\mathbf{v})) \cdot \nabla_{\mathbf{r}} y_\alpha, \quad (\text{A13})$$

where ψ_α and the projection operator \mathcal{P} are defined *mutatis mutandis* as in (26) and (27)

$$\psi_\nu = \frac{\partial f^{(0)}}{\partial y_\nu}, \quad \int d\mathbf{v} \mathbf{a}_\alpha(\mathbf{v}) \psi_\nu(\mathbf{v}) = \delta_{\alpha\nu}, \quad \mathcal{P}g(\mathbf{v}) = \psi_\nu(\mathbf{v}) \int d\mathbf{v} \mathbf{a}_\nu(\mathbf{v}) g(\mathbf{v}). \quad (\text{A14})$$

The gradients of y_α are arbitrary in (A10) so their coefficients give the desired equations for \mathbf{F}_α

$$\left(w \frac{\partial}{\partial e} + \mathcal{L}\right) \mathbf{F}_\alpha + \frac{\partial w}{\partial y_\alpha} \mathbf{F}_2 = -\mathcal{Q}(\mathbf{v} \psi_\alpha(\mathbf{v})), \quad \mathcal{Q} = 1 - \mathcal{P}. \quad (\text{A15})$$

This can be simplified further by noting that $w \propto (nT)^{3/2} n^{1/2}$ and using the scaling of (A4)

$$w \frac{\partial}{\partial e} = \frac{2w}{dn} \frac{\partial}{\partial T}, \quad \frac{\partial w}{\partial n} = \frac{1}{2} \frac{w}{n}, \quad \frac{\partial w}{\partial e} = \frac{2}{3n} \frac{\partial w}{\partial T} = \frac{w}{nT}, \quad (\text{A16})$$

to get

$$(\mathcal{L} - \lambda_\alpha) \mathbf{F}_\alpha - \delta_{\alpha 1} \xi \frac{3T}{4} \mathbf{F}_2 = -\mathcal{Q}(\mathbf{v} \psi_\alpha(\mathbf{v})), \quad (\text{A17})$$

$$\mathcal{L} \equiv \frac{1}{2} \xi \nabla_{\mathbf{v}} \cdot \mathbf{V} + L. \quad (\text{A18})$$

The constants λ_α

$$\lambda_\alpha \Leftrightarrow \left(0, \frac{1}{2} \xi, -\frac{1}{2} \xi\right) \quad (\text{A19})$$

are eigenvalues of the operator $\frac{1}{2} \xi \nabla_{\mathbf{v}} \cdot \mathbf{V} + \mathcal{L}$

$$(\mathcal{L} - \lambda_\alpha) \varphi_\alpha = 0 \quad (\text{A20})$$

The eigenfunctions φ_α are linear combinations of the set $\{\psi_\alpha\}$. Since the operator \mathcal{Q} projects orthogonal to this null space the Fredholm alternative is satisfied for these integral equations and their solutions exist.

The set of equations (A17) can be written in matrix form

$$(\mathcal{L}I - \bar{\Lambda})_{\alpha\beta} \mathbf{F}_\beta = -\mathcal{Q}(\mathbf{v} \psi_\alpha(\mathbf{v})) = -g_{\alpha\beta}^{-1} \mathcal{Q}(\gamma_\beta(\mathbf{v})), \quad (\text{A21})$$

with

$$\bar{\Lambda}_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta} - \frac{w}{2n} \delta_{\alpha 1} \delta_{\beta 2}, \quad (\text{A22})$$

where I is the identity matrix. Also, the fluxes γ_β are defined as in (33)

$$\gamma_\alpha(\mathbf{v}) = \mathbf{v} (\psi g)_\alpha = \mathbf{v} \frac{\partial f^{(0)}(\mathbf{v})}{\partial \chi_\alpha}. \quad (\text{A23})$$

Next, perform a similarity transformation with the symmetric matrix g to get the form

$$(\mathcal{L}I - \Lambda)_{\alpha\beta} (g\mathbf{F})_\beta = -\mathcal{Q}(\gamma_\alpha(\mathbf{v})), \quad \Lambda = g\bar{\Lambda}g^{-1}. \quad (\text{A24})$$

The solution (A1) can be given the representation

$$\delta f = f - f^{(0)} = - \left[\mathcal{Q} \int_0^\infty d\tau e^{-\mathcal{L}\tau} (e^{\Lambda\tau})_{\alpha\beta} \mathcal{Q} \gamma_\beta \right] \cdot \nabla_{\mathbf{r}} \chi_\alpha. \quad (\text{A25})$$

The irreversible fluxes are obtained from (29)

$$\mathbf{j}_\alpha^* = \int d\mathbf{v} \mathbf{b}_\alpha \mathcal{Q} \delta f = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_\alpha) e^{-\mathcal{L}\tau} (e^{\Lambda s} \mathcal{Q} \boldsymbol{\gamma})_\beta \cdot \nabla \chi_\beta \quad (\text{A26})$$

The transport matrix is given by (36) with the correlation functions [23]

$$C_{\alpha\beta}(t) = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_\alpha) e^{-\mathcal{L}\tau} (e^{\Lambda s} \mathcal{Q} \boldsymbol{\gamma})_\beta. \quad (\text{A27})$$

In particular, the correlation function determining the transport coefficient μ is

$$C_{21}(t) = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_2) e^{-\mathcal{L}\tau} (e^{\Lambda s} \mathcal{Q} \boldsymbol{\gamma})_1 \quad (\text{A28})$$

A more explicit form is obtained by direct solution to the equations for \mathbf{F}_1 and \mathbf{F}_2

$$\mathbf{F}_1 = \frac{3T}{2} \mathbf{F}_2 - \int_0^\infty d\tau e^{-(\mathcal{L}-\lambda_1)\tau} \mathcal{Q} \left(\mathbf{v} \left[\psi_1 - \xi \frac{3T}{4(\lambda_2 - \lambda_1)} \psi_2 \right] \right). \quad (\text{A29})$$

$$\mathbf{F}_2 = - \int_0^\infty d\tau e^{-(\mathcal{L}-\lambda_2)\tau} \mathcal{Q} (\mathbf{v} \psi_2(\mathbf{v})). \quad (\text{A30})$$

The coefficient μ is then

$$\mu = \int_0^\infty d\tau C_{21}(\tau) \frac{\partial \chi_1}{\partial n} \Big|_T = \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_2) \left(\mathbf{F}_1 + \mathbf{F}_2 \frac{\partial e}{\partial n} \Big|_T \right). \quad (\text{A31})$$

Substituting (A29) and (A30) leads after some rearrangement to

$$\begin{aligned} \mu = & - \int_0^\infty d\tau \int d\mathbf{v} (\mathcal{Q}^\dagger \mathbf{b}_2) \left\{ e^{-(\mathcal{L}-\lambda_1)\tau} \mathcal{Q} \mathbf{v} \frac{\partial f^{(0)}}{\partial n} \Big|_T \right. \\ & \left. - 2 \frac{\partial e}{\partial n} \Big|_T (e^{-(\mathcal{L}-\lambda_2)\tau} - e^{-(\mathcal{L}-\lambda_1)\tau}) \mathcal{Q} \left(\mathbf{v} \frac{\partial f^{(0)}}{\partial e} \Big|_n \right) \right\} \end{aligned} \quad (\text{A32})$$

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